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Subnormality in ω_1^2

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Abstract

A space X is said to be *subnormal* (= δ-*normal*) if every pair of disjoint closed sets can be separated by disjoint G_δ-sets. It is known that the product space $(\omega_1 + 1) \times \omega_1$ is neither normal nor subnormal, moreover the subspace $A \times B$ of ω_1^2 is not normal whenever A and B are disjoint stationary sets in ω_1. We will discuss on subnormality of subspaces of ω_1^2.

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All spaces considered in this paper are regular and T_1. A space X is said to be *subnormal* (= δ-*normal*) if every pair of disjoint closed sets can be separated by disjoint G_δ-sets, see [1,7]. It is well known that all subspaces of ordinals, more generally all GO-spaces, are shrinking, so normal and countably paracompact. But, as is well known, the product space $(\omega_1 + 1) \times \omega_1$ is countably paracompact but not normal. Indeed, first the product space $(\omega_1 + 1) \times \omega_1$ is the perfect preimage of the countably paracompact space ω_1, so it is countably paracompact. Second, the Pressing Down Lemma (abbreviated as PDL) shows that the diagonal $\Delta = \{\langle \alpha, \alpha \rangle \in (\omega_1 + 1) \times \omega_1\colon \alpha < \omega_1\}$ and the closed set $\{\omega_1\} \times \omega_1$ cannot be separated by disjoint open sets. Moreover similarly, we can show that these two disjoint closed subsets cannot be separated by disjoint G_δ sets, so $(\omega_1 + 1) \times \omega_1$ is not subnormal [5]. A space X is said to be *countably subparacompact* if every countable open cover has a σ-locally finite closed refinement, equivalently every countable open cover has a countable closed refinement. Note that countable subparacompactness implies subnormality, therefore $(\omega_1 + 1) \times \omega_1$ is, strangely, not countably subparacompact. On the other hand, it is known that all subspaces of two ordinals are always countably metacompact [4] and that $X = A \times B$ is neither normal nor countably paracompact

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whenever A and B are disjoint stationary sets in ω_1 [3]. So it is natural to ask whether the above space $X = A \times B$ is subnormal (or countably subparacompact) or not. In this paper, we will see that all subspaces of ω_1^2 are countably subparacompact, therefore subnormal.

For $A \subset \omega_1$, put $\mathrm{Lim}(A) = \{\alpha < \omega_1\colon \sup(A \cap \alpha) = \alpha\}$, where $\sup \emptyset = -1$, $\mathrm{Succ}(A) = A \setminus \mathrm{Lim}(A)$, $\mathrm{Lim} = \mathrm{Lim}(\omega_1)$ and $\mathrm{Succ} = \mathrm{Succ}(\omega_1)$. Observe that $\mathrm{Lim}(A)$ is closed and unbounded (cub) in ω_1 whenever A is unbounded in ω_1. For a cub set $C \subset \omega_1$ and $\alpha \in C$, put $p_C(\alpha) = \sup(C \cap \alpha)$. Observe that $p_C(\alpha) \in C \cup \{-1\}$, and $p_C(\alpha) = \alpha$ iff $\alpha \in \mathrm{Lim}(C)$, and $p_C(\alpha)$ is the immediate predecessor of α in $C \cup \{-1\}$ whenever $\alpha \in \mathrm{Succ}(C)$. It is easy to show that $\omega_1 \setminus C = \bigcup_{\alpha \in \mathrm{Succ}(C)} (p_C(\alpha), \alpha)$ and $\omega_1 \setminus \mathrm{Lim}(C) = \bigcup_{\alpha \in \mathrm{Succ}(C)} (p_C(\alpha), \alpha]$, where (α, β) and $(\alpha, \beta]$ denote the usual open and half open, respectively, interval.

Assume that a cub set C_α is defined for each $\alpha \in A$, where $A \subset \omega_1$. Then the diagonal intersection $\Delta_{\alpha \in A} C_\alpha = \{\beta \in \omega_1\colon \forall \alpha \in A \cap \beta (\beta \in C_\alpha)\}$ of C_α's, $\alpha \in A$, is a cub set in ω_1 (see [6, Lemma II 6.14]).

We use the following specific notation: Let $X \subset \omega_1^2$, $\alpha < \omega_1$ and $\beta < \omega_1$. Let $V_\alpha(X) = \{\beta < \omega_1\colon \langle \alpha, \beta \rangle \in X\}$, $H_\beta(X) = \{\alpha < \omega_1\colon \langle \alpha, \beta \rangle \in X\}$ and $\Delta(X) = \{\alpha < \omega_1\colon \langle \alpha, \alpha \rangle \in X\}$. For subsets C and D of ω_1, let $X_C = X \cap C \times \omega_1$, $X^D = X \cap \omega_1 \times D$ and $X_C^D = X \cap C \times D$.

Let $\mathcal{U}$ be an open cover of a space X. A collection $\mathcal{F} = \{F(U)\colon U \in \mathcal{U}\}$ indexed by $\mathcal{U}$ is said to be a *shrinking* (*subshrinking*) of $\mathcal{U}$ in X if $F(U) \subset U$ and $F(U)$ is closed (F_σ, respectively) in X for each $U \in \mathcal{U}$, and $\mathcal{F}$ covers X. A space is said to be *shrinking* (*subshrinking*, see [7]) if every open cover has a shrinking (subshrinking). Note that countable subparacompactness is equivalent to the assertion that every countable open cover has a subshrinking. Therefore subshrinking implies countable subparacompactness and countable subparacompactness implies subnormality.

Theorem A. *All subspaces of ω_1^2 are subshrinking.*

To prove this, we need several lemmas. The following is easy.

Lemma 1. *If X_n is a closed subshrinking subspace of a space X for each $n \in \omega$, then the subspace $\bigcup_{n \in \omega} X_n$ of X is also subshrinking.*

So we have:

Lemma 2. *$\alpha \times \omega_1$ and $\omega_1 \times \alpha$ are hereditarily subshrinking for each $\alpha < \omega_1$. In particular, for each subspace X of ω_1^2, $X_{[0,\alpha]}$ and $X^{[0,\alpha]}$ are subshrinking clopen subspaces of X for each $\alpha < \omega_1$.*

This lemma shows that, for each cub set $C \subset \omega_1$ and $X \subset \omega_1^2$,

$$X_{\omega_1 \setminus \mathrm{Lim}(C)} = \bigoplus_{\alpha \in \mathrm{Succ}(C)} X_{(p_C(\alpha), \alpha]} \quad \text{and} \quad X^{\omega_1 \setminus \mathrm{Lim}(C)} = \bigoplus_{\alpha \in \mathrm{Succ}(C)} X^{(p_C(\alpha), \alpha]}$$

are also subshrinking.

Let $X \subset \omega_1^2$, $Y = \{\langle \alpha, \beta \rangle \in X\colon \alpha \leqslant \beta\}$ and $Z = \{\langle \alpha, \beta \rangle \in X\colon \alpha \geqslant \beta\}$. Then X is the union of the two closed subspaces Y and Z. So by Lemma 1, to show the subshrinking property of X, it suffices to show that both Y and Z are subshrinking. Since the two cases are similar, we may assume $X \subset \{\langle\alpha, \beta\rangle \in \omega_1^2\colon\ \alpha \leqslant \beta\}$ and we will show X is subshrinking. The following is routine.

Lemma 3. *Let $\mathcal{G}$ be a collection of G_δ-sets of a space X. If there is a point-finite collection $\mathcal{U} = \{U(G)\colon\ G \in \mathcal{G}\}$ of open sets with $G \subset U(G)$, then $\bigcup \mathcal{G}$ is also a G_δ-set in X.*

Lemma 4. *Let $X \subset \{\langle\alpha, \beta\rangle \in \omega_1^2\colon\ \alpha \leqslant \beta\}$ be such that $X \cap C^2 = \emptyset$ for some cub set $C \subset \omega_1$. Then X is subshrinking.*

Proof. Let $\beta \in \mathrm{Succ}(C)$. Since $X^{(p_C(\beta),\beta)}$ is a countable open subspace of X and $X_C^{(p_C(\beta),\beta]} \subset X^{(p_C(\beta),\beta)}$, $X_C^{(p_C(\beta),\beta]}$ is G_δ in X. Moreover since $\{X^{(p_C(\beta),\beta)}\colon\ \beta \in \mathrm{Succ}(C)\}$ is a pairwise disjoint collection of open sets with $X_C^{(p_C(\beta),\beta]} \subset X^{(p_C(\beta),\beta)}$, by Lemma 3,

$$X_C = \bigoplus_{\beta \in \mathrm{Succ}(C)} X_C^{(p_C(\beta),\beta]}$$

is also G_δ in X. Say $X_C = \bigcap_{n\in\omega} V_n$, where V_n's are open in X. Since $X_{\omega_1 \setminus \mathrm{Lim}(C)}$ is subshrinking and $X \setminus V_n \subset X \setminus X_C = X_{\omega_1 \setminus C} \subset X_{\omega_1 \setminus \mathrm{Lim}(C)}$, $X \setminus V_n$'s are closed subshrinking subspaces of X. On the other hand, since $X^{\omega_1 \setminus \mathrm{Lim}(C)}$ is subshrinking and $X_C \subset X \setminus X^C = X^{\omega_1 \setminus C} \subset X^{\omega_1 \setminus \mathrm{Lim}(C)}$, X_C is a closed subshrinking subspace of X. Then X is covered by the countable collection $\{X_C\} \cup \{X \setminus V_n\colon\ n \in \omega\}$ of closed subshrinking subspaces of X. Therefore X is itself subshrinking. □

Lemma 5. *Let $X \subset \{\langle\alpha, \beta\rangle \in \omega_1^2\colon\ \alpha \leqslant \beta\}$ and $\alpha_0 < \omega_1$. Assume that there are a cub set $D \subset \omega_1$ with $X \cap \{\langle\alpha, \alpha\rangle\colon\ \alpha \in D\} = \emptyset$, an uncountable subset S of D and a function $g : S \to \omega_1$ such that, for each $\alpha \in S$,*

(1) $\alpha \leqslant g(\alpha)$,

(2) $g(\alpha') < \alpha$ *for each* $\alpha' \in S \cap \alpha$.

Then

$$Z(\alpha_0, D, S, g) = \bigcup_{\alpha \in S} X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)}$$

is an open F_σ subset of X and there is a cub set $C \subset \omega_1$ such that $X \cap C^2 \subset Z(\alpha_0, D, S, g)$.

Proof. Let $Z = Z(\alpha_0, D, S, g)$. It is evident that Z is open in X. For each $\gamma \in \mathrm{Lim}$, fix a strictly increasing cofinal sequence $\{\gamma(n)\colon\ n \in \omega\}$ in γ. For each $\gamma \in \mathrm{Lim}(S)$ and $n \in \omega$, let $\alpha(\gamma) = \min\{\alpha \in S\colon\ \gamma \leqslant \alpha\}$ and $V_n(\gamma) = X_{(\gamma(n),\gamma]}^{(\gamma, g(\alpha(\gamma))]}$. Note that $V_n(\gamma)$ is clopen in X.

Claim 1. *The collection $\{(\gamma, g(\alpha(\gamma))]\colon\ \gamma \in \mathrm{Lim}(S)\}$ is pairwise disjoint.*

Proof. Let $\gamma', \gamma \in \mathrm{Lim}(S)$ with $\gamma' < \gamma$. It follows from $\gamma' < \gamma \in \mathrm{Lim}(S)$ that there are $\alpha', \alpha \in S$ with $\gamma' < \alpha' < \alpha < \gamma$. By the minimality of $\alpha(\gamma')$ and $\alpha(\gamma)$, we have $\gamma' \leqslant \alpha(\gamma') \leqslant \alpha' < \alpha < \gamma \leqslant \alpha(\gamma)$. Moreover by (1), (2) and $\alpha \in S$, we have $\gamma' \leqslant \alpha(\gamma') \leqslant g(\alpha(\gamma')) < \alpha < \gamma \leqslant \alpha(\gamma) \leqslant g(\alpha(\gamma))$. Therefore $(\gamma', g(\alpha(\gamma'))] \cap (\gamma, g(\alpha(\gamma))] = \emptyset$. □

So note that $\{X^{(\gamma,g(\alpha(\gamma))]}\colon \gamma \in \mathrm{Lim}(S)\}$ is a pairwise disjoint collection of clopen sets and $V_n(\gamma) \subset X^{(\gamma,g(\alpha(\gamma))]}$ for each $\gamma \in \mathrm{Lim}(S)$ and $n \in \omega$. Let $V_n = \bigcup_{\gamma \in \mathrm{Lim}(S)} V_n(\gamma)$ and $F_n = Z \setminus V_n$ for each $n \in \omega$.

Claim 2. *F_n is closed in X for each $n \in \omega$.*

Proof. Let $\langle \mu, \nu \rangle \in X \setminus F_n$. We will find a neighborhood of $\langle \mu, \nu \rangle$ disjoint from F_n. Since V_n is an open set disjoint from F_n, we may assume $\langle \mu, \nu \rangle \notin Z \cup V_n$. When $\mu \leqslant \alpha_0$, $X_{[0,\alpha_0]}$ is a neighborhood of $\langle \mu, \nu \rangle$ disjoint from F_n. So let $\alpha_0 < \mu$ and take the minimal $\gamma \in \mathrm{Lim}(S)$ with $\mu \leqslant \gamma$. Assume $\mu = \gamma$. Then since $\mathrm{Lim}(S) \subset D$ and X is disjoint from $\{\langle \alpha, \alpha \rangle\colon \alpha \in D\}$, we have $\mu = \gamma < \nu$. If $\nu \leqslant g(\alpha(\gamma))$, then

$$\langle \mu, \nu \rangle = \langle \gamma, \nu \rangle \in X^{(\gamma,g(\alpha(\gamma))]}_{(\gamma(n),\gamma]} = V_n(\gamma) \subset V_n,$$

a contradiction. If $g(\alpha(\gamma)) < \nu$, then $\langle \mu, \nu \rangle \in X^{(g(\alpha(\gamma)),\omega_1)}_{(\alpha_0,\alpha(\gamma)]} \subset Z$, a contradiction. Therefore we have $\mu < \gamma$. Take the minimal $\alpha \in S$ with $\mu \leqslant \alpha$. It follows from $\mu < \gamma \in \mathrm{Lim}(S)$ that $\mu \leqslant \alpha < \gamma$. By the minimality of γ, we have $\alpha \notin \mathrm{Lim}(S)$. It follows from $\langle \mu, \nu \rangle \notin Z \supset X^{(g(\alpha),\omega_1)}_{(\alpha_0,\alpha]}$ that $\nu \leqslant g(\alpha)$. On the other hand, by the minimality of α, we have $S \cap \alpha \subset \mu$, so $\sup(S \cap \alpha) \leqslant \mu$. Assume $\sup(S \cap \alpha) = \mu$. Then we have $S \cap \alpha = S \cap \mu$. Therefore $\sup(S \cap \mu) = \mu$, so $\mu \in \mathrm{Lim}(S)$. This contradicts the minimality of γ and $\mu < \gamma$. So we have $\mu_0 = \sup(S \cap \alpha) < \mu$. We will show $X^{[0,\nu]}_{(\mu_0,\mu]} \cap Z = \emptyset$. Indeed let $\alpha' \in S$. If $\alpha \leqslant \alpha'$, then by $\nu \leqslant g(\alpha) \leqslant g(\alpha')$, we have

$$X^{[0,\nu]}_{(\mu_0,\mu]} \cap X^{(g(\alpha'),\omega_1)}_{(\alpha_0,\alpha']} = \emptyset.$$

If $\alpha' < \alpha$, then by $\alpha' \leqslant \mu_0$, we have

$$X^{[0,\nu]}_{(\mu_0,\mu]} \cap X^{(g(\alpha'),\omega_1)}_{(\alpha_0,\alpha']} = \emptyset.$$

Finally, by $F_n \subset Z$, $X^{[0,\nu]}_{(\mu_0,\mu]}$ is a neighborhood of $\langle \mu, \nu \rangle$ disjoint from F_n. Therefore F_n is closed. □

Claim 3. $Z = \bigcup_{n \in \omega} F_n$.

Proof. $\bigcup_{n \in \omega} F_n \subset Z$ is evident. Let $\langle \mu, \nu \rangle \in Z$. Since $V_n = \bigcup_{\gamma \in \mathrm{Lim}(S)} V_n(\gamma) \subset \bigcup_{\gamma \in \mathrm{Lim}(S)} X^{(\gamma,g(\alpha(\gamma))]}$ for each $n \in \omega$, we may assume $\langle \mu, \nu \rangle \in X^{(\gamma,g(\alpha(\gamma))]}$ for some $\gamma \in \mathrm{Lim}(S)$. Then $\gamma < \nu \leqslant g(\alpha(\gamma))$. It follows from $\langle \mu, \nu \rangle \in Z$ that $\langle \mu, \nu \rangle \in X^{(g(\alpha),\omega_1)}_{(\alpha_0,\alpha]}$ for some $\alpha \in S$, in particular, $g(\alpha) < \nu$ and $\mu \leqslant \alpha$. Assume $\gamma \leqslant \alpha$. Then, by the minimality of $\alpha(\gamma)$, we have $\alpha(\gamma) \leqslant \alpha$. Therefore $\nu \leqslant g(\alpha(\gamma)) \leqslant g(\alpha)$, a contradiction. So we have $\alpha < \gamma$. Since $\alpha < \gamma \in \mathrm{Lim}(S) \subset \mathrm{Lim}$, there is $n \in \omega$ with $\alpha \leqslant \gamma(n) < \gamma$. By $\mu \leqslant \alpha$, we have $\langle \mu, \nu \rangle \notin V_n(\gamma)$. By Claim 1 and $\langle \mu, \nu \rangle \in X^{(\gamma,g(\alpha(\gamma))]}$, $\langle \mu, \nu \rangle \notin V_n(\gamma')$ for each $\gamma' \in \mathrm{Lim}(S)$ with $\gamma' \neq \gamma$. Therefore we have $\langle \mu, \nu \rangle \notin V_n$, so $\langle \mu, \nu \rangle \in F_n$. □

Finally we will find a cub set $C \subset \omega_1$ such that $X \cap C^2 \subset Z$. For each $\alpha < \omega_1$ with $\alpha_0 < \alpha$, take the minimal $\gamma \in S$ with $\alpha \leqslant \gamma$ and set $h(\alpha) = g(\gamma)$. Then by the definition of Z, $X^{(h(\alpha),\omega_1)}_{\{\alpha\}} \subset Z$. Let $C = (\alpha_0, \omega_1) \cap D \cap \Delta_{\alpha \in (\alpha_0,\omega_1)}(h(\alpha), \omega_1)$. Then C is cub.

Let $\langle\alpha,\beta\rangle \in X \cap C^2$. Since $C \subset D$ and $X \cap \{\langle\alpha,\alpha\rangle\colon \alpha \in D\} = \emptyset$, we have $\alpha < \beta$, so $\alpha \in (\alpha_0,\omega_1) \cap \beta$. On the other hand, by $\beta \in \Delta_{\alpha\in(\alpha_0,\omega_1)}(h(\alpha),\omega_1)$, we have $\beta \in (h(\alpha),\omega_1)$. Therefore by $\alpha_0 < \alpha$, $\langle\alpha,\beta\rangle \in X_{\{\alpha\}}^{(h(\alpha),\omega_1)} \subset Z$, and so $X \cap C^2 \subset Z$. □

Proof of Theorem A. Assume $X \subset \{\langle\alpha,\beta\rangle \in \omega_1^2\colon \alpha \leqslant \beta\}$. Let $\mathcal{U}$ be an open cover of X.

Case 1. $\Delta(X) = \{\alpha < \omega_1\colon \langle\alpha,\alpha\rangle \in X\}$ is stationary in ω_1.

In this case, for each $\alpha \in \Delta(X)$, fix $f(\alpha) < \alpha$ and $U(\alpha) \in \mathcal{U}$ such that $X_{(f(\alpha),\alpha]}^{(f(\alpha),\alpha]} \subset U(\alpha)$. By the PDL, there are $\alpha_0 < \omega_1$ and a stationary set $S \subset \Delta(X)$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. For each pair $\alpha,\beta \in S$, define $\alpha \simeq \beta$ by $U(\alpha) = U(\beta)$. Then obviously $\simeq$ is an equivalence relation on S. For each equivalence class E in the quotient $S/{\simeq}$, define $U(E) = U(\alpha)$ for some (equivalently, arbitrary) $\alpha \in E$. Note that

$$X_{(\alpha_0,\alpha]}^{(\alpha_0,\alpha]} \subset U(E) \quad \text{for each } \alpha \in E. \tag{$*$}$$

There are two subcases to consider.

Case 1.1. There is $E_0 \in S/{\simeq}$ such that E_0 is unbounded in ω_1.

By $(*)$, we have $X_{(\alpha_0,\omega_1)}^{(\alpha_0,\omega_1)} \subset U(E_0)$. Note that

$$X = X_{[0,\alpha_0]} \oplus X_{(\alpha_0,\omega_1)}^{(\alpha_0,\omega_1)}$$

and $X_{[0,\alpha_0]}$ is subshrinking by Lemma 2. So we can find a subshrinking $\mathcal{H} = \{H(U)\colon U \in \mathcal{U}\}$ of $\{U \cap X_{[0,\alpha_0]}\colon U \in \mathcal{U}\}$ in $X_{[0,\alpha_0]}$. For each $U \in \mathcal{U}$, let

$$F(U) = \begin{cases} H(U) \cup X_{(\alpha_0,\omega_1)}^{(\alpha_0,\omega_1)}, & \text{if } U = U(E_0),\\ H(U), & \text{otherwise.}\end{cases}$$

Then $\mathcal{F} = \{F(U)\colon U \in \mathcal{U}\}$ is a subshrinking of $\mathcal{U}$ in X.

Case 1.2. E is bounded for each $E \in S/{\simeq}$.

By induction on $\gamma < \omega_1$, we can find a strictly increasing sequence $\{\alpha(\gamma)\colon \gamma < \omega_1\} \subset S$ and a sequence $\{E(\gamma)\colon \gamma < \omega_1\} \subset S/{\simeq}$ as follows. Assume that $\gamma < \omega_1$, $\{\alpha(\gamma')\colon \gamma' < \gamma\}$ and $\{E(\gamma')\colon \gamma' < \gamma\}$ are already defined. Pick $\alpha(\gamma) \in S$ with $\alpha(\gamma) > \sup(\bigcup_{\gamma'<\gamma} E(\gamma')) + \gamma$ and $E(\gamma) \in S/{\simeq}$ with $\alpha(\gamma) \in E(\gamma)$. Then by the construction, all $E(\gamma)$'s are distinct and $X_{(\alpha_0,\alpha(\gamma)]}^{(\alpha_0,\alpha(\gamma)]} \subset U(E(\gamma))$ for each $\gamma < \omega_1$.

Let, as above, $\mathcal{H} = \{H(U)\colon U \in \mathcal{U}\}$ be a subshrinking of $\{U \cap X_{[0,\alpha_0]}\colon U \in \mathcal{U}\}$ in $X_{[0,\alpha_0]}$.

For each $U \in \mathcal{U}$, let

$$F(U) = \begin{cases} H(U) \cup X_{(\alpha_0,\alpha(\gamma)]}^{(\alpha_0,\alpha(\gamma)]}, & \text{if } U = U(E(\gamma)) \text{ for some } \gamma < \omega_1,\\ H(U), & \text{otherwise.}\end{cases}$$

Then $\mathcal{F} = \{F(U)\colon U \in \mathcal{U}\}$ is a subshrinking of $\mathcal{U}$ in X.

Case 2. $\Delta(X)$ is not stationary in ω_1.

Let $A = \{\alpha < \omega_1\colon V_\alpha(X) \text{ is stationary in } \omega_1\}$ and let D be a cub set disjoint from $\Delta(X)$.

Case 2.1. A is not stationary in ω_1.

Let C' be a cub set with $C' \subset D$ and $C' \cap A = \emptyset$. For each $\alpha \in C'$, fix a cub set C_α disjoint from $V_\alpha(X)$. Let $C = C' \cap \Delta_{\alpha\in C'} C_\alpha$. Assume $\langle\alpha,\beta\rangle \in X \cap C^2$. It follows from $C \subset C' \subset D$ that $\alpha < \beta$, so $\alpha \in C \cap \beta \subset C' \cap \beta$. Moreover by $\beta \in C \subset \Delta_{\alpha\in C'} C_\alpha$, we have $\beta \in C_\alpha$, so $\beta \notin V_\alpha(X)$. This contradicts $\langle\alpha,\beta\rangle \in X$. Therefore $X \cap C^2 = \emptyset$. Then, by Lemma 4, X is subshrinking.

Case 2.2. A is stationary in ω_1.

Let $\alpha \in A \cap D$ and $\beta \in V_\alpha(X)$. Since $\mathcal{U}$ is an open cover of X, fix $f(\alpha,\beta) < \alpha$, $g(\alpha,\beta) < \beta$ and $U(\alpha,\beta) \in \mathcal{U}$ such that

$$X^{(g(\alpha,\beta),\beta]}_{(f(\alpha,\beta),\alpha]} \subset U(\alpha,\beta).$$

By $\alpha \in D$, we have $\alpha < \beta$, so we may assume $\alpha \leqslant g(\alpha,\beta)$. Since $V_\alpha(X)$ is stationary and $|\alpha| \leqslant \omega$, by applying the PDL, we can find a stationary set $T_\alpha \subset V_\alpha(X)$, $f(\alpha) < \alpha$ and $g(\alpha) < \omega_1$ such that $f(\alpha,\beta) = f(\alpha)$ and $\alpha \leqslant g(\alpha,\beta) = g(\alpha)$ for each $\beta \in T_\alpha$. For convenience, let $g(\alpha) = 0$ for each $\alpha \in \omega_1 \setminus (A \cap D)$. Then $D' = \{\alpha < \omega_1 \colon \forall \alpha' < \alpha (g(\alpha') < \alpha)\}$ is cub. Since $A \cap D \cap D'$ is stationary, applying the PDL again, we find a stationary set $S \subset A \cap D \cap D'$ and $\alpha_0 < \omega_1$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. Then, for each $\alpha \in S$ and $\beta \in T_\alpha$, we have $X^{(g(\alpha),\beta]}_{(\alpha_0,\alpha]} \subset U(\alpha,\beta)$. Now note that α_0, D, S and g satisfy all assumptions of Lemma 5. Let $H = \bigcup_{\alpha \in S} \{\alpha\} \times T_\alpha$. For $\langle \alpha', \beta' \rangle, \langle \alpha, \beta \rangle \in H$, define $\langle \alpha', \beta' \rangle \simeq \langle \alpha, \beta \rangle$ by $U(\alpha',\beta') = U(\alpha,\beta)$. For each equivalence class E in the quotient $H/\simeq$, define $U(E) = U(\alpha,\beta)$ for some (equivalently, arbitrary) $\langle \alpha,\beta \rangle \in E$. Then

$$\bigcup_{\langle \alpha,\beta \rangle \in E} X^{(g(\alpha),\beta]}_{(\alpha_0,\alpha]} \subset U(E) \tag{+}$$

and the $U(E)$'s are distinct. For each $\alpha \in S$ and $E \in H/\simeq$, let $j(E,\alpha) = \sup V_\alpha(E)$, $S(E) = \{\alpha \in S \colon j(E,\alpha) = \omega_1\}$ and $k(E) = \sup S(E)$.

Case 2.2.1. There is $E_0 \in H/\simeq$ such that $k(E_0) = \omega_1$.

Note that $S(E_0)$ is unbounded in ω_1 and $S(E_0) \subset S \subset D \cap D' \subset D$. By Lemma 5, $Z = Z(\alpha_0, D, S(E_0), g)$ is an open F_σ set in X, $(X \setminus Z) \cap C^2 = \emptyset$ for some cub set C and $Z = Z(\alpha_0, D, S(E_0), g) = \bigcup_{\alpha \in S(E_0)} X^{(g(\alpha),\omega_1)}_{(\alpha_0,\alpha]} \subset \bigcup_{\langle \alpha,\beta \rangle \in E_0} X^{(g(\alpha),\beta]}_{(\alpha_0,\alpha]} \subset U(E_0)$. By Lemma 4, $X \setminus Z$ is a closed subshrinking subspace of X. So there is a subshrinking $\mathcal{H} = \{H(U) \colon U \in \mathcal{U}\}$ of $\{U \cap (X \setminus Z) \colon U \in \mathcal{U}\}$ in $X \setminus Z$.

For each $U \in \mathcal{U}$, let

$$F(U) = \begin{cases} H(U) \cup Z, & \text{if } U = U(E_0), \\ H(U), & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) \colon U \in \mathcal{U}\}$ is a subshrinking of $\mathcal{U}$ in X.

Case 2.2.2. $k(E) < \omega_1$ for each $E \in H/\simeq$.

There are two subcases.

Case 2.2.2.1. $\sup\{k(E) \colon E \in H/\simeq\} = \omega_1$.

In this case, by induction, we can find a strictly increasing sequence $\{\alpha(\gamma) \colon \gamma < \omega_1\} \subset S$ and a sequence $\{E(\gamma) \colon \gamma < \omega_1\} \subset H/\simeq$ such that $\sup(\bigcup_{\gamma' < \gamma} S(E(\gamma))) + \gamma < \alpha(\gamma) \in S(E(\gamma))$. Let $S' = \{\alpha(\gamma) \colon \gamma < \omega_1\}$. Then $S' \subset S \subset D$, S' is unbounded in ω_1 and

$$X^{(g(\alpha(\gamma)),\omega_1)}_{(\alpha_0,\alpha(\gamma)]} = \bigcup_{\beta \in V_{\alpha(\gamma)}(E(\gamma))} X^{(g(\alpha(\gamma)),\beta]}_{(\alpha_0,\alpha(\gamma)]} \subset U\bigl(E(\gamma)\bigr)$$

for each $\gamma < \omega_1$. By Lemma 5,

$$Z = Z(\alpha_0, D, S', g) = \bigcup_{\alpha \in S'} X^{(g(\alpha),\omega_1)}_{(\alpha_0,\alpha]} = \bigcup_{\gamma < \omega_1} X^{(g(\alpha(\gamma)),\omega_1)}_{(\alpha_0,\alpha(\gamma)]}$$

is an open F_σ set in X and $(X\backslash Z)\cap C^2=\emptyset$ for some cub set C. By Lemma 4, there is a subshrinking $\mathcal{H}=\{H(U)\colon U\in\mathcal{U}\}$ of $\{U\cap(X\backslash Z)\colon U\in\mathcal{U}\}$ in $X\backslash Z$. For each $U\in\mathcal{U}$, let

$$F(U)=\begin{cases} H(U)\cup X^{(g(\alpha),\omega_1)]}_{(\alpha_0,\alpha(\gamma)]}, & \text{if } U=U(E(\gamma)) \text{ for some } \gamma<\omega_1,\\ H(U), & \text{otherwise.}\end{cases}$$

Then $\mathcal{F}=\{F(U)\colon U\in\mathcal{U}\}$ is a subshrinking of $\mathcal{U}$ in X.

Case 2.2.2.2. $\sup\{k(E)\colon E\in H/\simeq\}<\omega_1$.

Fix $\alpha_1<\omega_1$ with $\sup\{k(E)\colon E\in H/\simeq\}+\alpha_0<\alpha_1$. Note that $\sup V_\alpha(E)<\omega_1$ for each $\alpha\in S$ with $\alpha_1<\alpha$ and $E\in H/\simeq$. Let $S'=\{\alpha\in S\colon \alpha_1<\alpha\}$ and $H'=\bigcup_{\alpha\in S'}\{\alpha\}\times T_\alpha$. Consider the co-lexicographic order $\prec$ on H', that is, $\langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle$ iff $\delta'<\delta$ or ($\delta'=\delta$ and $\gamma'<\gamma$) for each $\langle\gamma',\delta'\rangle,\langle\gamma,\delta\rangle\in H'$. Since $H'\subset X\subset\{\langle\alpha,\beta\rangle\in\omega_1^2\colon \alpha\leqslant\beta\}$, for each $\langle\gamma,\delta\rangle\in H'$, the $\prec$-initial segument $\{\langle\gamma',\delta'\rangle\in H'\colon \langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle\}$ of $\langle\gamma,\delta\rangle$ is contained in the countable set $\{\langle\alpha,\beta\rangle\in\omega_1^2\colon \alpha\leqslant\beta\leqslant\delta\}$. So by $|H'|=\omega_1$, the order type of the well-ordered set $\langle H',\prec\rangle$ is exactly ω_1. By $\prec$-induction on H', we will construct a strictly $\prec$-increasing sequence $\{\beta(\gamma,\delta)\colon \langle\gamma,\delta\rangle\in H'\}\subset\omega_1$ and a sequence $\{E(\gamma,\delta)\colon \langle\gamma,\delta\rangle\in H'\}\subset H/\simeq$ such that

$$\bigl\langle\gamma,\beta(\gamma,\delta)\bigr\rangle\in E(\gamma,\delta), \tag{1}$$

$$\begin{aligned}&\sup\bigl\{j(E(\gamma',\delta'),\gamma)\colon \langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle\bigr\}\\ &\quad+\sup\bigl\{\beta(\gamma',\delta')\colon \langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle\bigr\}+\delta<\beta(\gamma,\delta)\in T_\gamma.\end{aligned} \tag{2}$$

Assume that $\beta(\gamma',\delta')$ and $E(\gamma',\delta')$ have been defined for each $\langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle$, where $\langle\gamma,\delta\rangle\in H'$. It follows from $\alpha_1<\gamma$ that $j(E,\gamma)<\omega_1$ for each $E\in H/\simeq$. So, since the $\prec$-initial segment of $\langle\gamma,\delta\rangle$ is countable and T_γ is stationary in ω_1, we can find $\beta(\gamma,\delta)\in T_\gamma$ with $\sup\{j(E(\gamma',\delta'),\gamma)\colon \langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle\}+\sup\{\beta(\gamma',\delta')\colon \langle\gamma',\delta'\rangle\prec\langle\gamma,\delta\rangle\}+\delta<\beta(\gamma,\delta)$. Then take $E(\gamma,\delta)\in H/\simeq$ with $\langle\gamma,\beta(\gamma,\delta)\rangle\in E(\gamma,\delta)$. By the construction, members of $\{E(\gamma,\delta)\colon \langle\gamma,\delta\rangle\in H'\}$ are distinct and

$$\{\beta(\gamma,\delta)\colon \delta\in T_\gamma\} \text{ is unbounded in } \omega_1 \text{ for each } \gamma\in S'. \tag{3}$$

Therefore by (3),

$$X^{(g(\gamma),\omega_1)}_{(\alpha_0,\gamma]}=\bigcup_{\delta\in T_\gamma}X^{(g(\gamma),\beta(\gamma,\delta)]}_{(\alpha_0,\gamma]}. \tag{4}$$

Moreover by Lemma 5 and (4),

$$\begin{aligned}Z\;=\;Z\bigl(\alpha_0,D,S',g\bigr)&=\bigcup_{\gamma\in S'}X^{(g(\gamma),\omega_1)}_{(\alpha_0,\gamma]}\\ &=\bigcup_{\gamma\in S'}\Bigl(\bigcup_{\delta\in T_\gamma}X^{(g(\gamma),\beta(\gamma,\delta)]}_{(\alpha_0,\gamma]}\Bigr)=\bigcup_{\langle\gamma,\delta\rangle\in H'}X^{(g(\gamma),\beta(\gamma,\delta)]}_{(\alpha_0,\gamma]}\end{aligned}$$

is an open F_σ set in X and $(X\backslash Z)\cap C^2=\emptyset$ for some cub set C. Note that

$$\bigl\{X^{(g(\gamma),\beta(\gamma,\delta)]}_{(\alpha_0,\gamma]}\colon \langle\gamma,\delta\rangle\in H'\bigr\}$$

is a collection of clopen set whose union is exactly Z and that by (+), $X_{(\alpha_0,\gamma]}^{(g(\gamma),\beta(\gamma,\delta)]} \subset U(E(\gamma,\delta))$ for each $\langle\gamma,\delta\rangle \in H'$. Let, by Lemma 4, $\mathcal{H} = \{H(U)\colon U \in \mathcal{U}\}$ be a subshrinking of $\{U \cap (X\setminus Z)\colon U \in \mathcal{U}\}$ in $X\setminus Z$.

For each $U \in \mathcal{U}$, let

$$F(U) = \begin{cases} H(U) \cup X_{(\alpha_0,\gamma]}^{(g(\gamma),\beta(\gamma,\delta)]}, & \text{if } U = U(E(\gamma,\delta)) \text{ for some } \langle\gamma,\delta\rangle \in H', \\ H(U), & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U)\colon U \in \mathcal{U}\}$ is a subshrinking of $\mathcal{U}$ in X. The proof of Theorem A is complete. □

In the rest of this paper, we consider collectionwise subnormality of subspaces of ω_1^2. A space X is *collectionwise subnormal* (abbreviated as CWSN), see [7], if for every discrete collection $\mathcal{F}$ of closed sets, there is a sequence $\{\mathcal{G}_n\colon n \in \omega\}$ of collections of open sets, where $\mathcal{G}_n$ is represented as $\{G_n(F)\colon F \in \mathcal{F}\}$ with $F \subset G_n(F)$, such that for each $x \in X$, there is $n \in \omega$ with $|\{F \in \mathcal{F}\colon x \in G_n(F)\}| \leqslant 1$. In this situation, $\{\mathcal{G}_n\colon n \in \omega\}$ is said to be a θ-expansion of $\mathcal{F}$. Moreover, a space X is collectionwise δ-normal (CWδN), see [1], if every discrete collection $\mathcal{F}$ of closed sets can be separated by G_δ-sets, that is, there is a pairwise disjoint collection $\mathcal{G} = \{G(F)\colon F \in \mathcal{F}\}$ of G_δ-sets with $F \subset G(F)$. It is easy to verify that CWSN implies CWδN. The following is known.

Proposition 6 [2]. *Every discrete collection $\mathcal{F}$ of closed sets in a subnormal space X with $|\mathcal{F}| \leqslant 2^\omega$ is separated by G_δ-sets.*

So, by $|\omega_1^2| \leqslant \omega_1 \leqslant 2^\omega$ and Theorem A, we have:

Proposition 7. *All subspaces of ω_1^2 are CWδN.*

But the author does not know whether CWδN implies CWSN or not, so hereafter we present a direct proof of the following theorem.

Theorem B. *All subspaces of ω_1^2 are CWSN.*

The proof of Theorem B is somewhat similar to that of Theorem A. It is straightforward to show:

Lemma 1′. *If X_n is a closed CWSN subspace of a space X for each $n \in \omega$, then the subspace $\bigcup_{n\in\omega} X_n$ of X is also CWSN.*

Applying Lemma 1′, we can similarly show:

Lemma 2′. *$\alpha \times \omega_1$ and $\omega_1 \times \alpha$ are hereditarily CWSN for each $\alpha < \omega_1$. In particular, for each subspace X of ω_1^2, $X_{[0,\alpha]}$ and $X^{[0,\alpha]}$ are CWSN clopen subspaces of X for each $\alpha < \omega_1$.*

Lemma 4′. *Let $X \subset \{\langle\alpha,\beta\rangle \in \omega_1^2\colon \alpha \leqslant \beta\}$ such that $X \cap C^2 = \emptyset$ for some cub set $C \subset \omega_1$. Then X is CWSN.*

Proof of Theorem B. Let $X \subset \{\langle\alpha,\beta\rangle \in \omega_1^2\colon \alpha \leqslant \beta\}$. It suffices to show that X is CWSN. Let $\mathcal{F}$ be a discrete collection of closed sets in X.

Case 1. $\Delta(X)$ is stationary in ω_1.

For each $\alpha \in \Delta(X)$, fix $f(\alpha) < \alpha$ such that

$$\bigl|\bigl\{F \in \mathcal{F}\colon X_{(f(\alpha),\alpha]}^{(f(\alpha),\alpha]} \cap F \neq \emptyset\bigr\}\bigr| \leqslant 1.$$

Then by the PDL, there are $\alpha_0 < \omega_1$ and a stationary set $S \subset \Delta(X)$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. Observe that

$$\bigl|\bigl\{F \in \mathcal{F}\colon X_{(\alpha_0,\omega_1)}^{(\alpha_0,\omega_1)} \cap F \neq \emptyset\bigr\}\bigr| \leqslant 1.$$

Since $X = X_{[0,\alpha_0]} \oplus X_{(\alpha_0,\omega_1)}^{(\alpha_0,\omega_1)}$ and $X_{[0,\alpha_0]}$ is CWSN by Lemma 2′, we can easily construct a θ-expansion of $\mathcal{F}$.

Case 2. $\Delta(X)$ is not stationary in ω_1.

Let $A = \{\alpha < \omega_1\colon V_\alpha(X)$ is stationary in $\omega_1\}$ and D be a cub set disjoint from $\Delta(X)$.

Case 2.1. A is not stationary in ω_1.

In this case, as in case 2.1 in the proof of Theorem A, $X \cap C^2 = \emptyset$ for some cub set C. Then apply Lemma 4′.

Case 2.2. A is stationary in ω_1.

Let $A_0 = \{\alpha \in A \cap D\colon V_\alpha(\bigcup\mathcal{F})$ is unbounded in $\omega_1\}$. Since $\mathcal{F}$ is discrete, applying the PDL, for each $\alpha \in A \cap D$, we can find $f(\alpha) < \alpha$ and $g(\alpha) < \omega_1$ with $\alpha \leqslant g(\alpha)$ such that:

(1) if $\alpha \in A_0$, then $|\{F \in \mathcal{F}\colon X_{(f(\alpha),\alpha]}^{(g(\alpha),\omega_1)} \cap F \neq \emptyset\}| = 1$,

(2) if $\alpha \in (A \cap D)\setminus A_0$, then $\{F \in \mathcal{F}\colon X_{(f(\alpha),\alpha]}^{(g(\alpha),\omega_1)} \cap F \neq \emptyset\} = \emptyset$.

Again applying the PDL to $A \cap D$ as in case 2.2 in the proof of Theorem A, we can find $\alpha_0 < \omega_1$ and a stationary set $S \subset A \cap D$ such that, for each $\alpha \in S$, $f(\alpha) = \alpha_0$, $g(\alpha') < \alpha$ for each $\alpha' \in S \cap \alpha$. Then observe that

(1′) if $\alpha \in S \cap A_0$, then $|\{F \in \mathcal{F}\colon X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)} \cap F \neq \emptyset\}| = 1$,

(2′) if $\alpha \in S\setminus A_0$, then $\{F \in \mathcal{F}\colon X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)} \cap F \neq \emptyset\} = \emptyset$.

Let $Z = \bigcup_{\alpha\in S} X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)}$ and $\mathcal{F}_0 = \{F \in \mathcal{F}\colon Z \cap F \neq \emptyset\}$.

Claim. $|\mathcal{F}_0| \leqslant 1$.

Proof. Assume that there are $F', F \in \mathcal{F}_0$ with $F' \neq F$. Then there are $\alpha', \alpha \in S$ such that $X_{(\alpha_0,\alpha']}^{(g(\alpha'),\omega_1)} \cap F' \neq \emptyset$ and $X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)} \cap F \neq \emptyset$. By (1′) and (2′), we have $\alpha', \alpha \in S \cap A_0$ and $\alpha' \neq \alpha$. We may assume $\alpha' < \alpha$. Since $\alpha' \in A_0$ and $\mathcal{F}$ is discrete, by (1′), $V_{\alpha'}(F')$ is unbounded in ω_1. Take $\beta \in V_{\alpha'}(F')$ with $\beta > g(\alpha)$. Then $\langle\alpha',\beta\rangle \in X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)} \cap F'$. Therefore $X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)} \cap F' \neq \emptyset$ and $X_{(\alpha_0,\alpha]}^{(g(\alpha),\omega_1)} \cap F \neq \emptyset$. By $\alpha \in S \cap A_0$ and $F' \neq F$, this contradicts (1′). This completes the proof of claim. □

By Lemma 5, Z is an open F_σ set of X and $(X\setminus Z) \cap C^2 = \emptyset$ for some cub set C. By Lemma 4′, $Y = X\setminus Z$ is a closed G_δ CWSN subspace of X, say $Y = \bigcap_{n\in\omega} G_n$, where G_n is open in X. Let $\{\mathcal{U}_n\colon n \in \omega\}$ be a θ-expansion of $\{F \cap Y\colon F \in \mathcal{F}\}$ in Y, say $\mathcal{U}_n = \{U_n(F)\colon F \in \mathcal{F}\}$ with $F \cap Y \subset U_n(F)$ and $U_n(F)$ is open in Y.

For each $F \in \mathcal{F}$ and $n \in \omega$, let

$$V_n(F) = \begin{cases} U_n(F) \cup Z, & \text{if } F \in \mathcal{F}_0, \\ U_n(F) \cup (G_n \setminus Y), & \text{otherwise.} \end{cases}$$

Then $V_n(F)$'s are open in X and $F \subset V_n(F)$. Set $\mathcal{V}_n = \{V_n(F)\colon F \in \mathcal{F}\}$ for each $n \in \omega$. To show that $\{\mathcal{V}_n\colon n \in \omega\}$ is a desired θ-expansion of $\mathcal{F}$ in X, let $x \in X$. If $x \in Z$, then there is $n \in \omega$ such that $x \notin G_n$, so $x \notin V_n(F)$ whenever $F \in \mathcal{F} \setminus \mathcal{F}_0$. If $x \in Y = X \setminus Z$, then for some $n \in \omega$, $x \in U_n(F)$ for at most one $F \in \mathcal{F}$. So $x \in V_n(F)$ for at most one $F \in \mathcal{F}$. The proof is complete. $\square$

The author conjectures that the answer of the following problem is, of course, "yes". But it seems to be somewhat complicated to handle the induction.

Problem. Are all subspaces of ω_1^n subnormal for each $n \in \omega$?

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